

# Mathematical Induction

## Part One

Everybody - do the wave!

# The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that?
  - Someone (me!) started everyone off.
  - Once the person before you did the wave, you did the wave.

Let  $P$  be some predicate. The ***principle of mathematical induction*** states that if

If it starts true...

$P(0)$  is true

...and it stays true...

and

$\forall k \in \mathbb{N}. (P(k) \rightarrow P(k+1))$

then

$\forall n \in \mathbb{N}. P(n)$

...then it's always true.

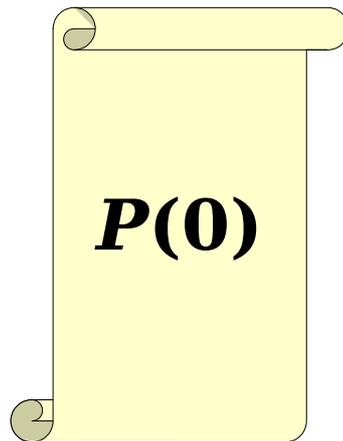
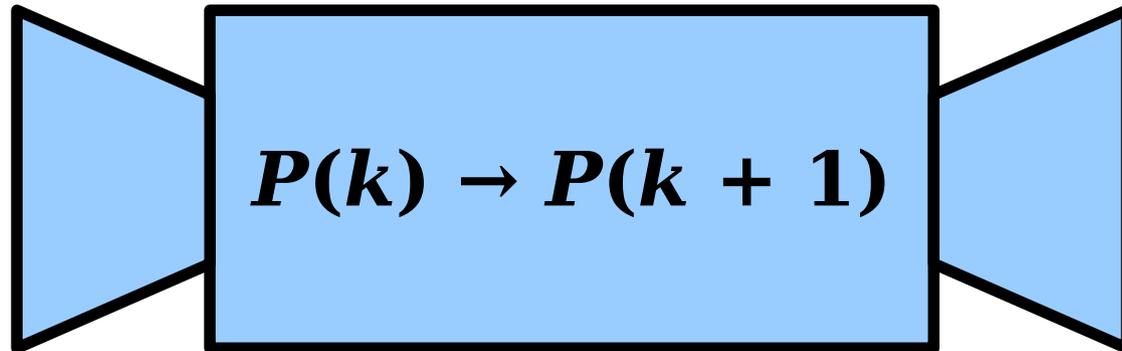
# Induction, Intuitively

**$P(0)$**

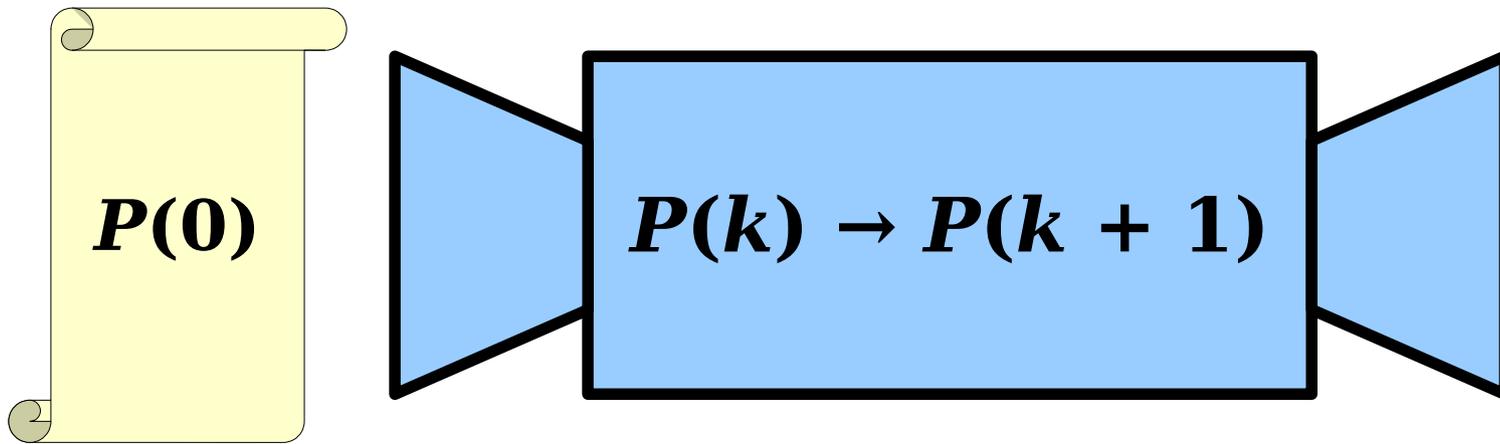
**$\forall k \in \mathbb{N}. (P(k) \rightarrow P(k+1))$**

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- ...

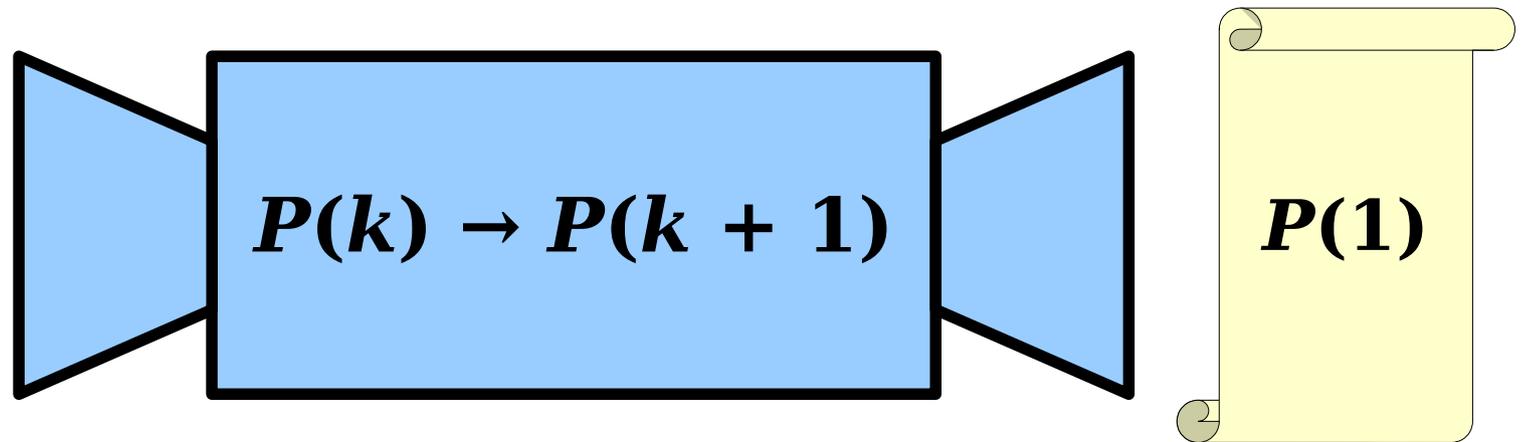
# Why Induction Works



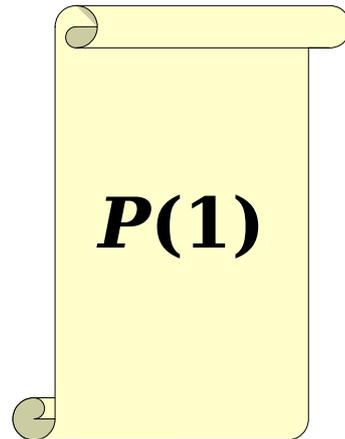
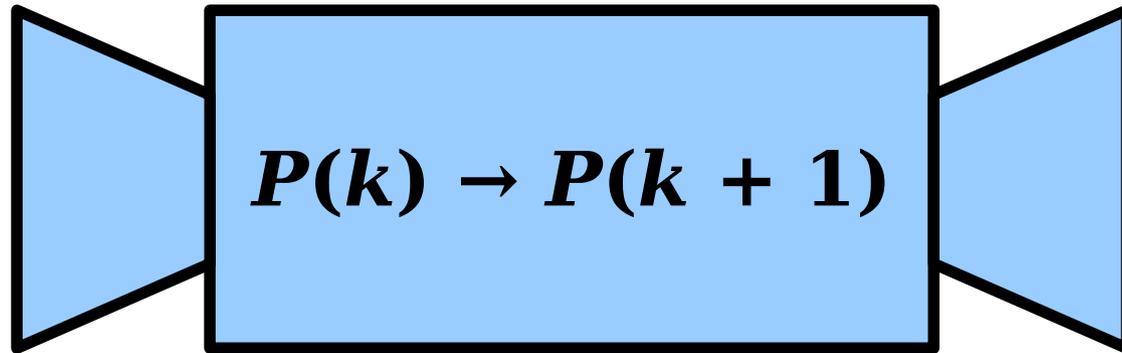
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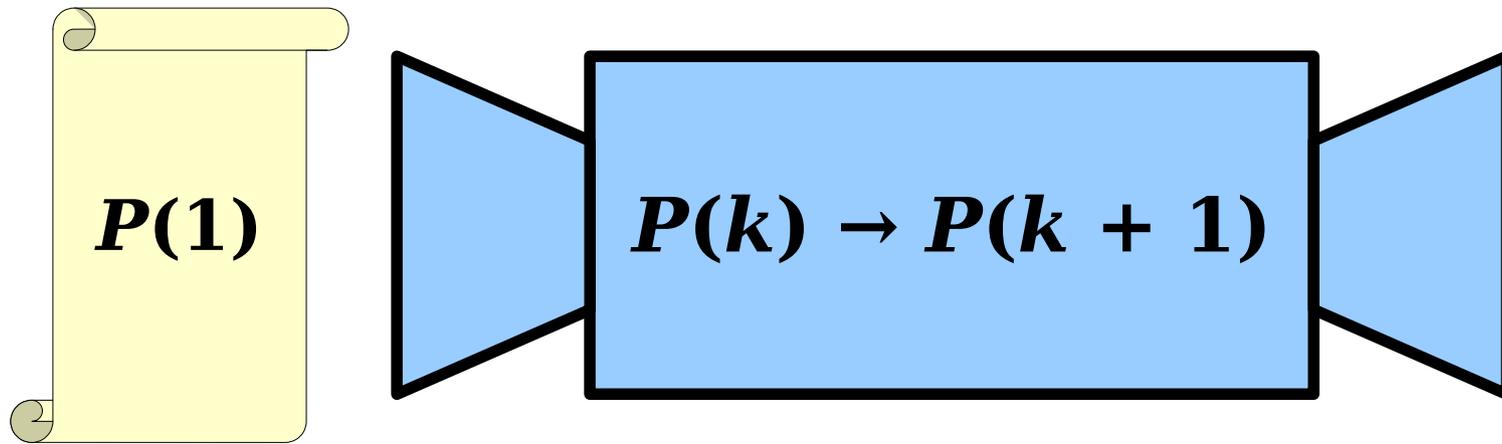
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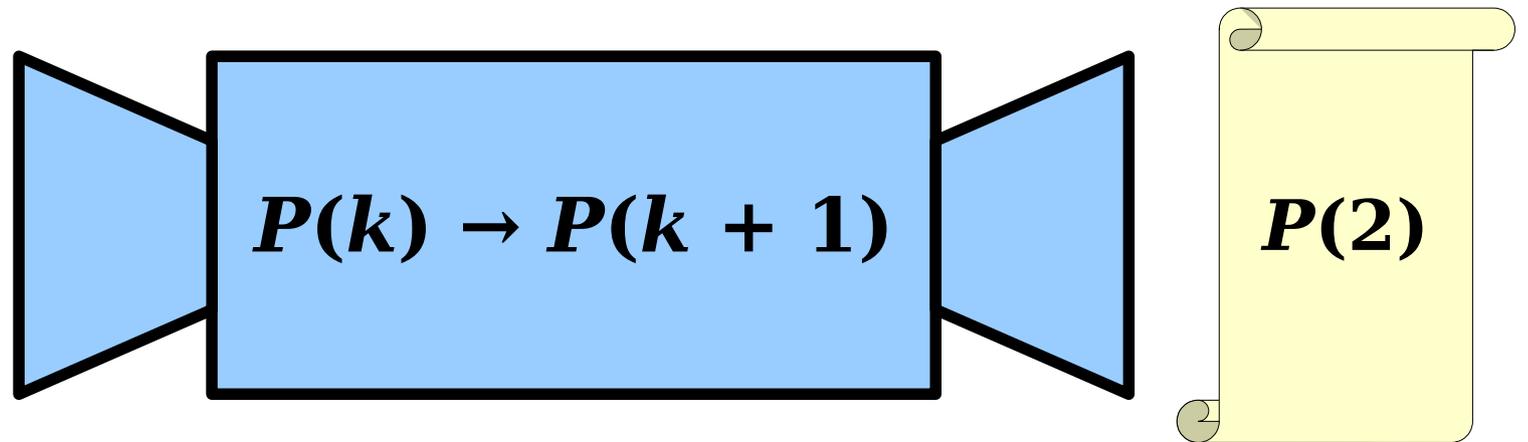
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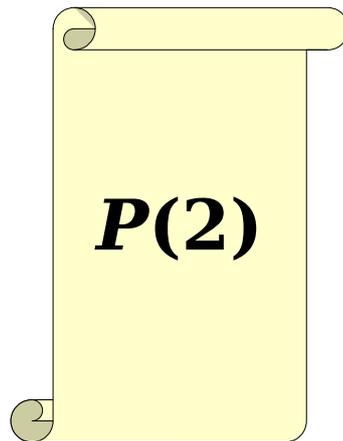
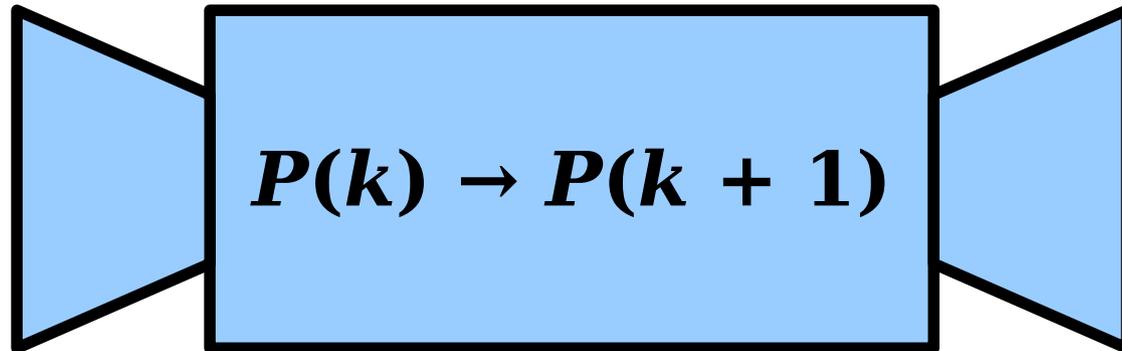
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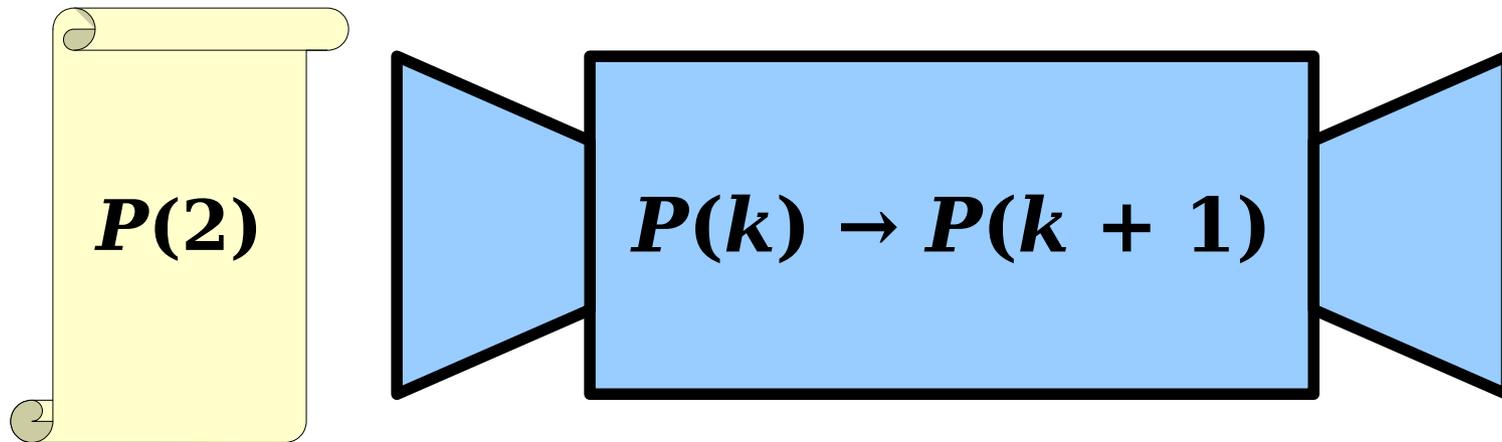
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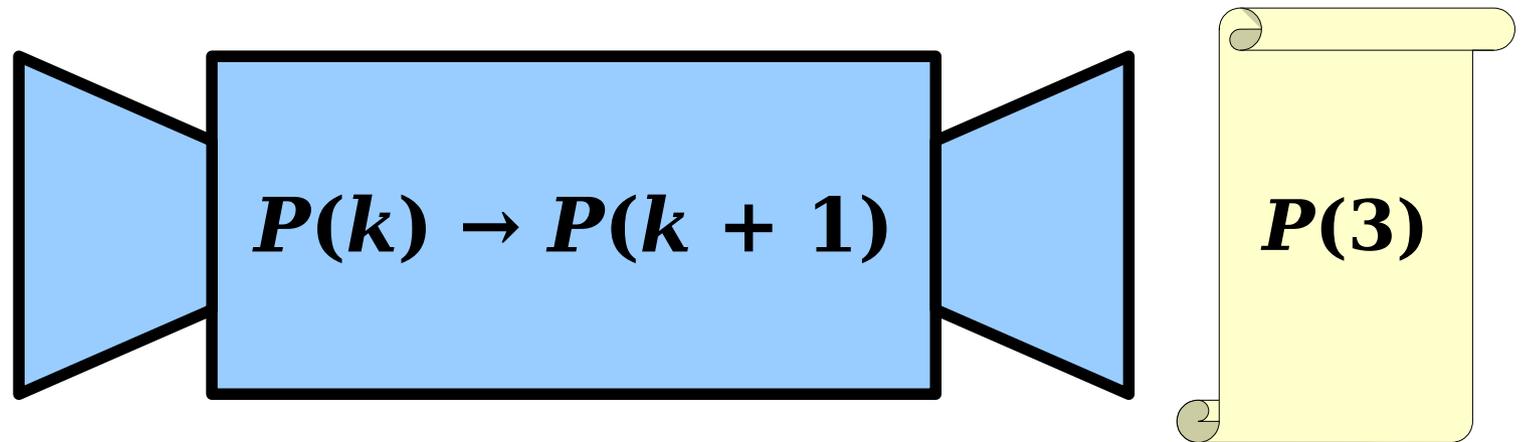
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# Proof by Induction

- A ***proof by induction*** is a way to use the principle of mathematical induction to show that some result is true for all natural numbers  $n$ .
- In a proof by induction, there are three steps:
  - Prove that  $P(0)$  is true.
    - This is called the ***basis*** or the ***base case***.
  - Prove that if  $P(k)$  is true, then  $P(k+1)$  is true.
    - This is called the ***inductive step***.
    - The assumption that  $P(k)$  is true is called the ***inductive hypothesis***.
  - Conclude, by induction, that  $P(n)$  is true for all  $n \in \mathbb{N}$ .

# Some Sums

$$2^0$$

$$2^0 + 2^1$$

$$2^0 + 2^1 + 2^2$$

$$2^0 + 2^1 + 2^2 + 2^3$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4$$

$$2^0 = 1$$

$$2^0 + 2^1 = 1 + 2 = 3$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31$$

$$2^0 = 1 = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers  $n$ , then tell them we're going to prove it by induction.

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Here, we state what  $P(0)$  actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

**“If  $P(k)$  is true, then  $P(k+1)$  is true.”**

So we ask the reader to pick some  $k$ , assume that  $P(k)$  is true, then try to prove  $P(k+1)$ .

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Here, we explicitly state  $P(k+1)$ , which is what we want to prove. Now, we can use any proof technique we want to prove it.

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Here, we'll use our **inductive hypothesis** (the assumption that  $P(k)$  is true) to simplify a complex expression. This is a common theme in inductive proofs.

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$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \quad (\text{via (1)}) \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore,  $P(k + 1)$  is true, completing the induction.

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For the inductive step, assume that for some arbitrary  $k \in \mathbb{N}$  that  $P(k)$  holds, meaning that

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Therefore,  $P(k + 1)$  is true, completing the induction. ■

# A Quick Aside

- This result helps explain the range of numbers that can be stored in an **int**.
- If you have an unsigned 32-bit integer, the largest value you can store is given by  $1 + 2 + 4 + 8 + \dots + 2^{31} = 2^{32} - 1$ .
- This formula for sums of powers of two has many other uses as well. You'll see one on Wednesday.

# Structuring a Proof by Induction

- Define some predicate  $P$  that you'll show, by induction, is true for all natural numbers.
- Prove the base case:
  - State that you're going to prove that  $P(0)$  is true, then go prove it.
- Prove the inductive step:
  - Say that you're assuming  $P(k)$  for some arbitrary natural number  $k$ , then write out exactly what that means.
  - Say that you're going to prove  $P(k+1)$ , then write out exactly what that means.
  - Prove that  $P(k+1)$  using any proof technique you'd like!
- This is a rather verbose way of writing inductive proofs. As we get more experience with induction, we'll start leaving out some details from our proofs.

# The Counterfeit Coin Problem

# Problem Statement

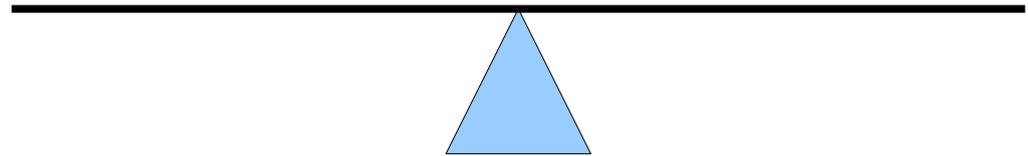
- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

How?

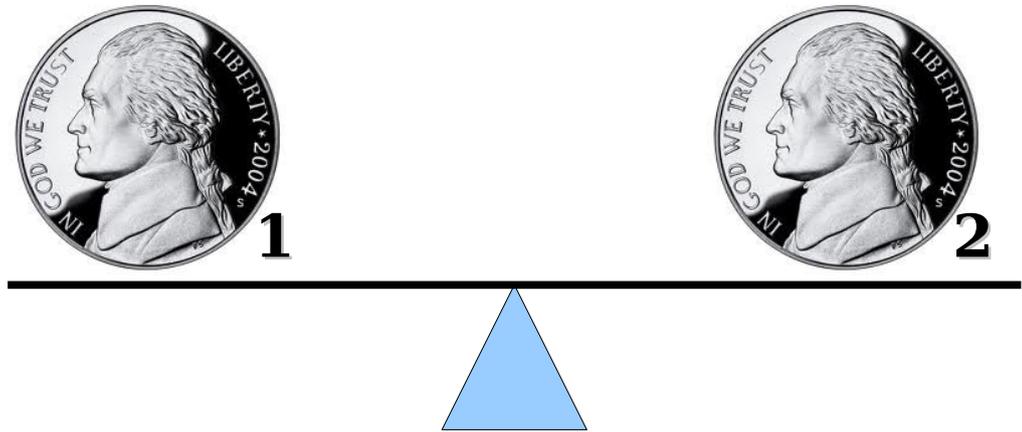
Answer at

<https://cs103.stanford.edu/pollev>

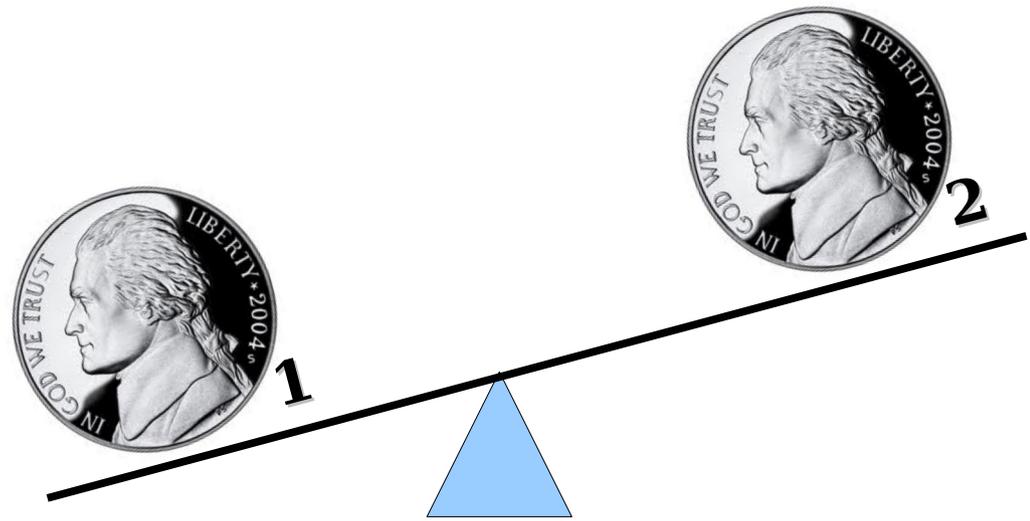
# Finding the Counterfeit Coin



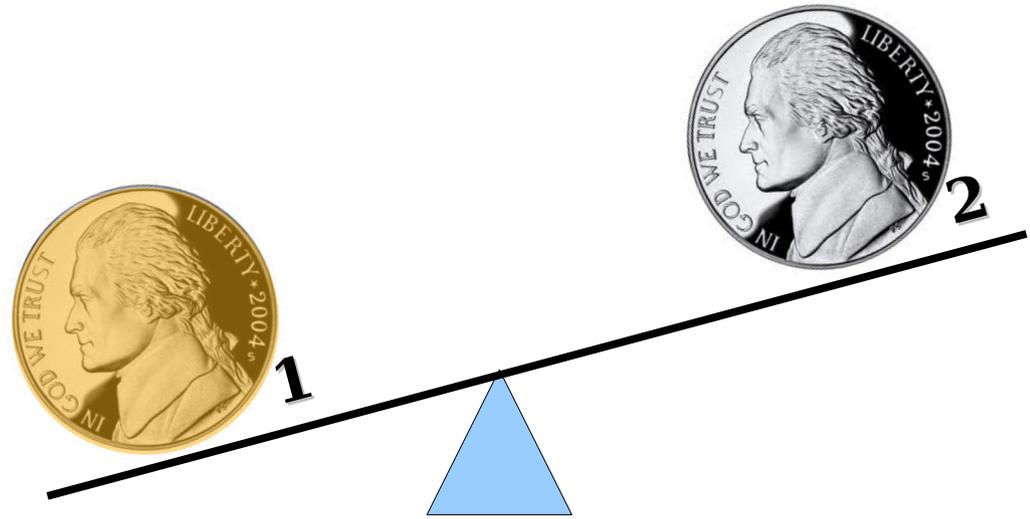
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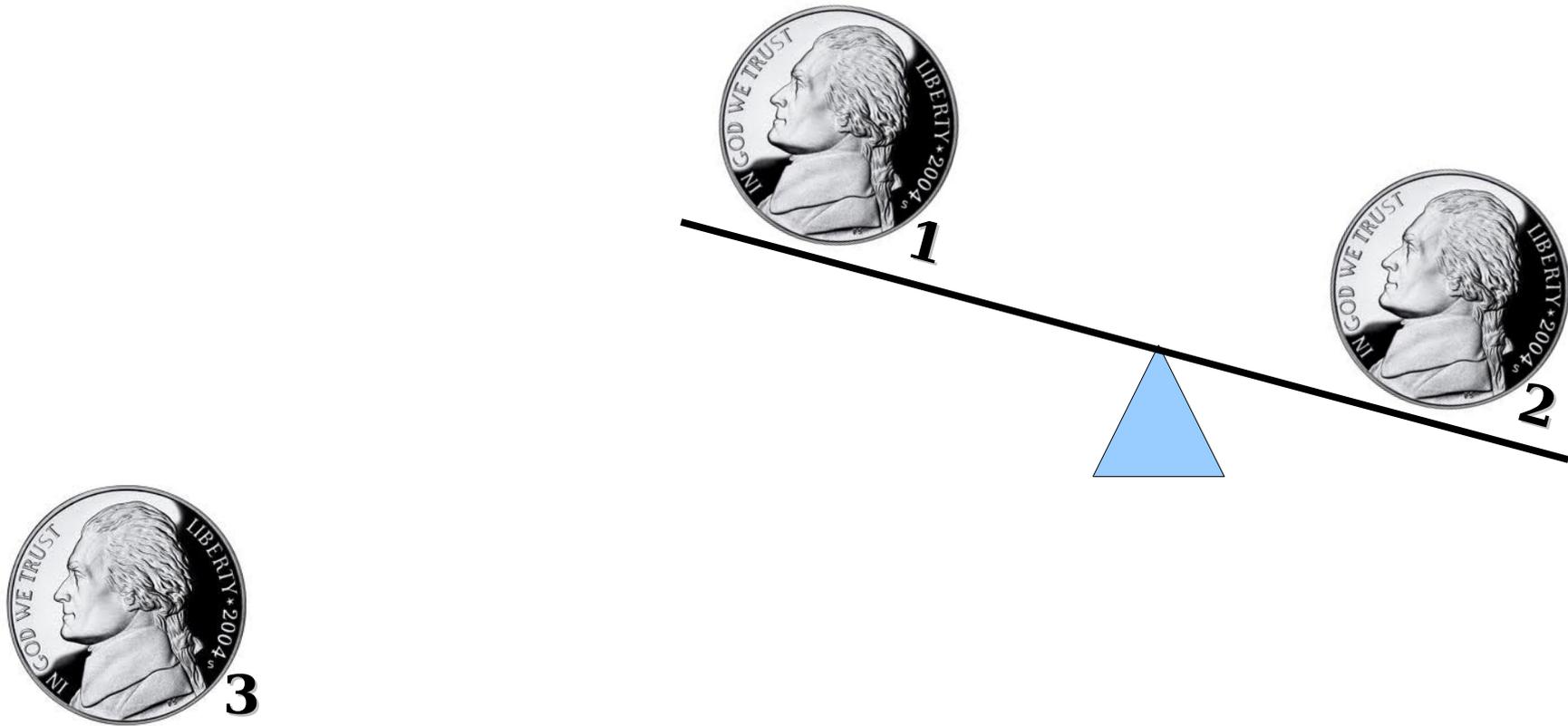
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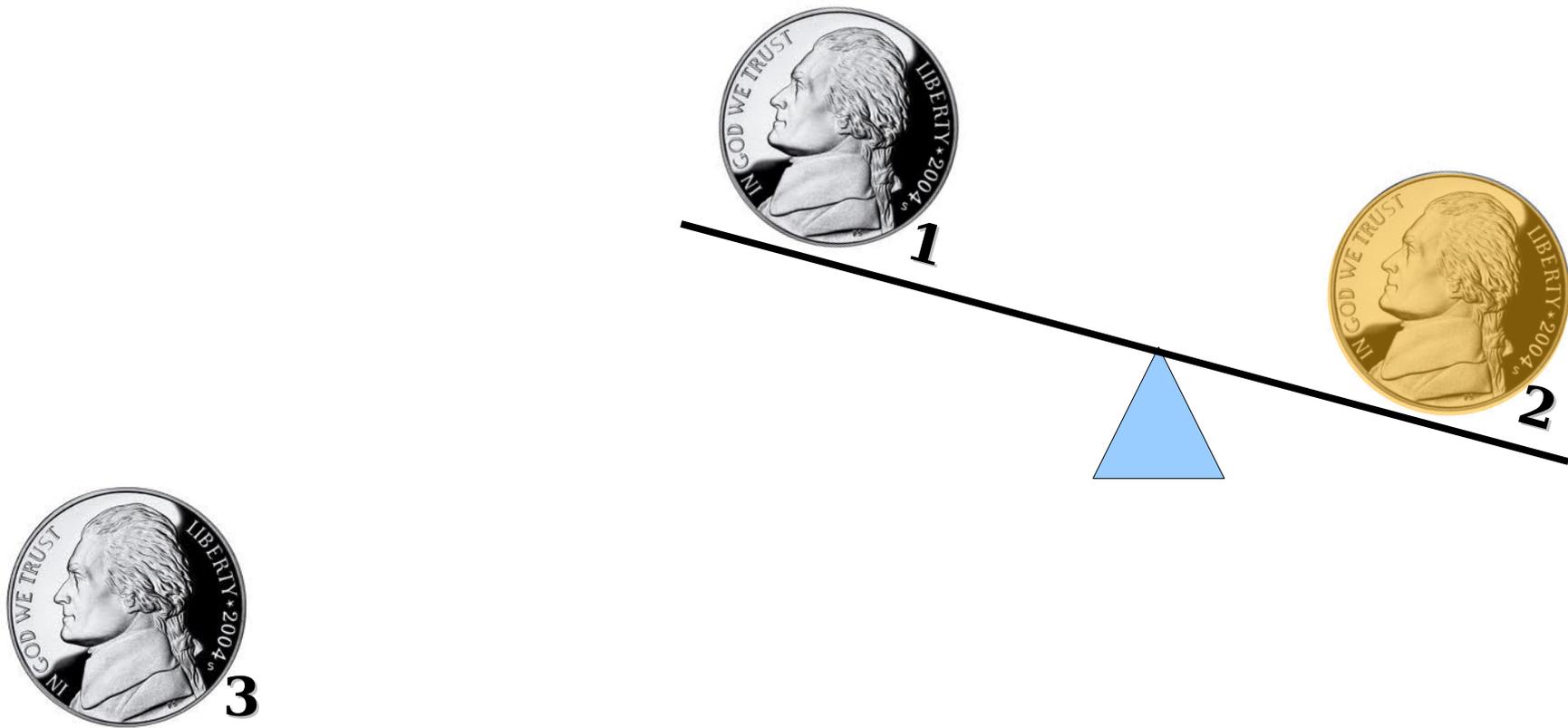
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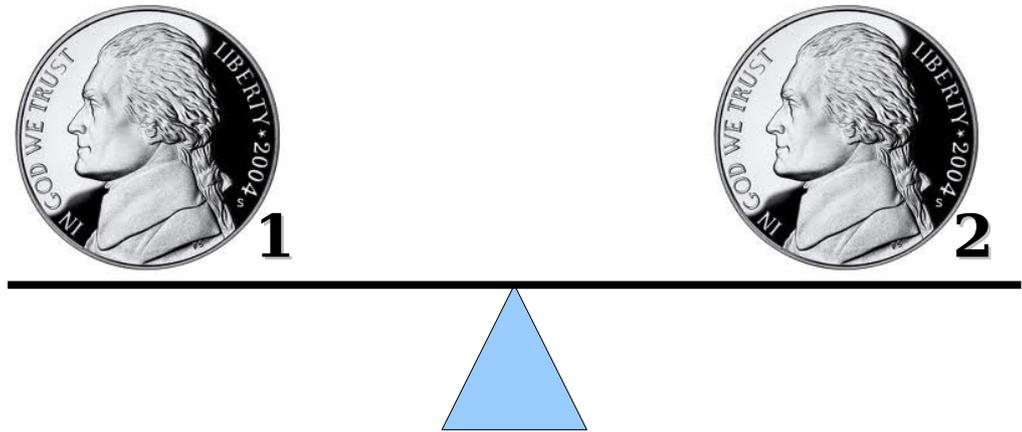
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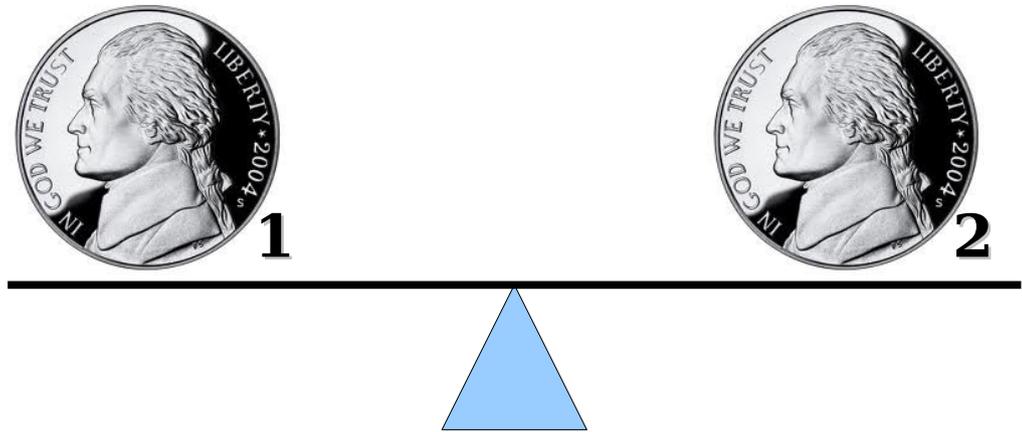
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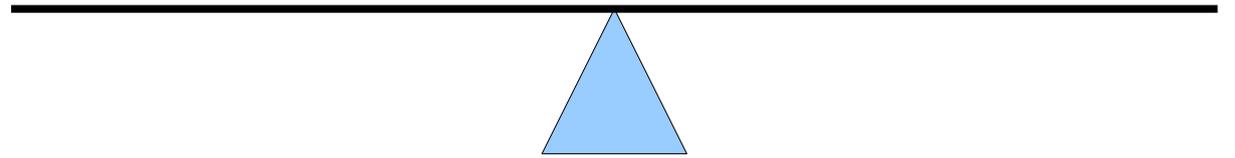
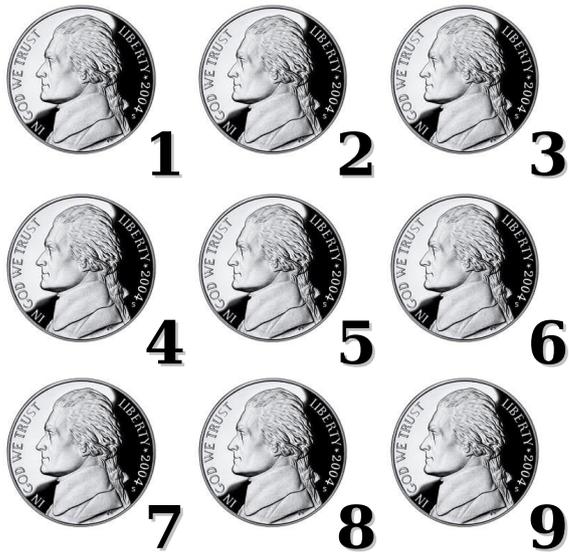
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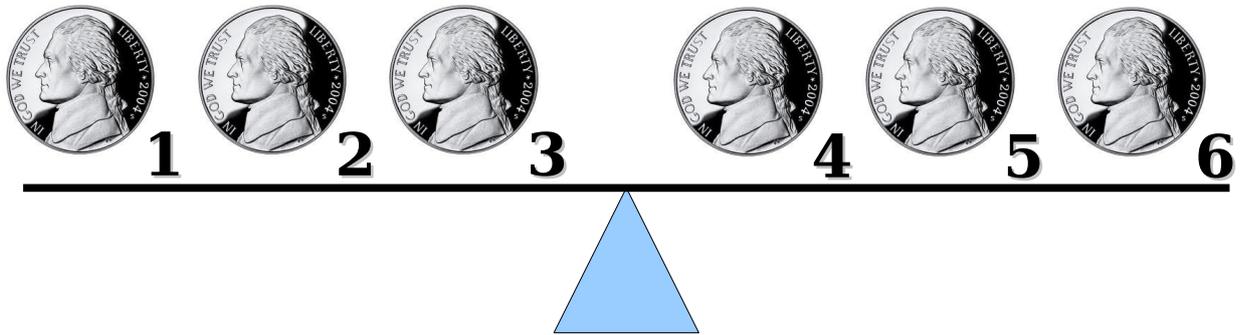
# A Harder Problem

- You are given a set of *nine* seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only *two* weighings on the balance, find the counterfeit coin.

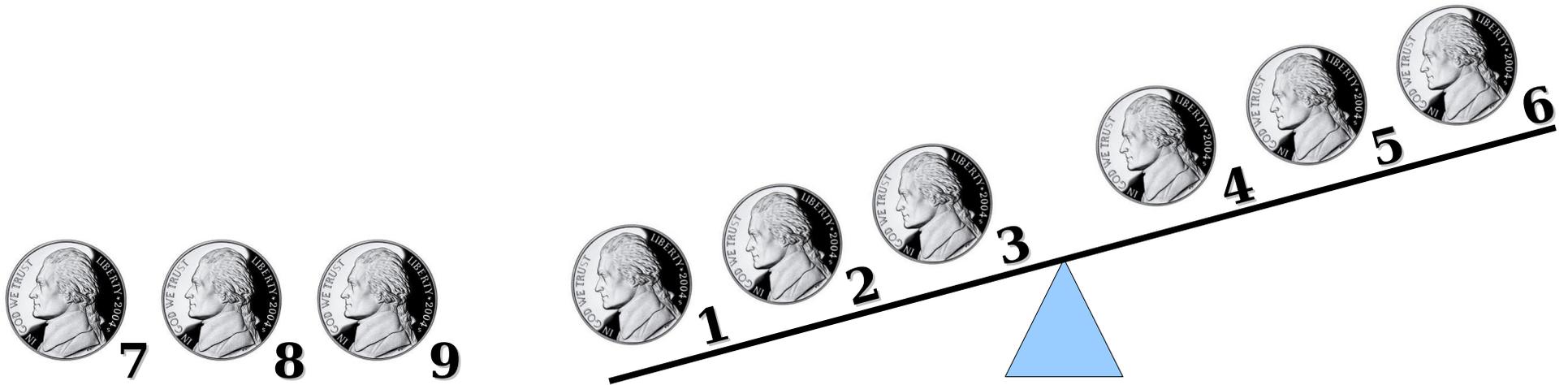
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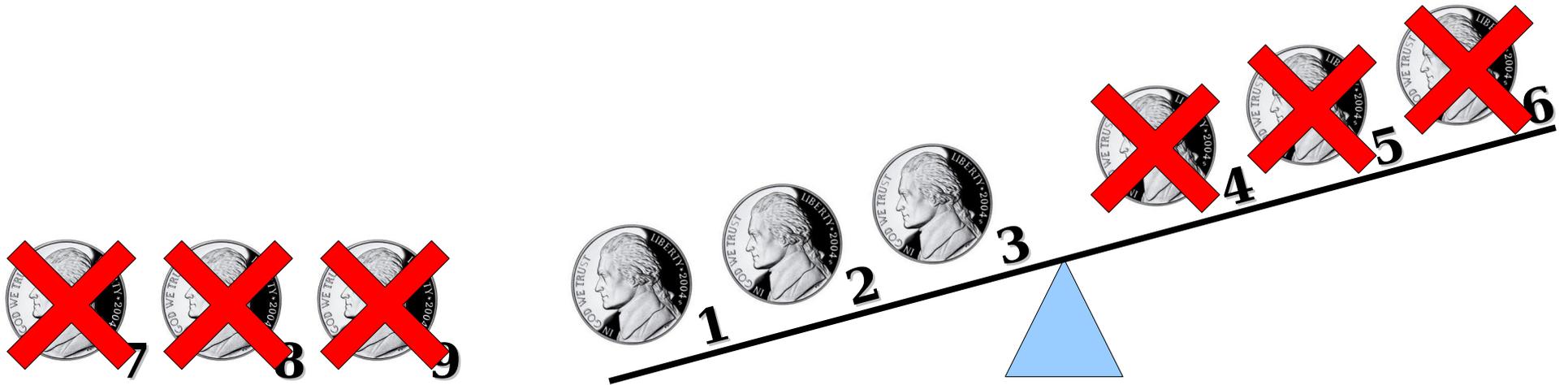
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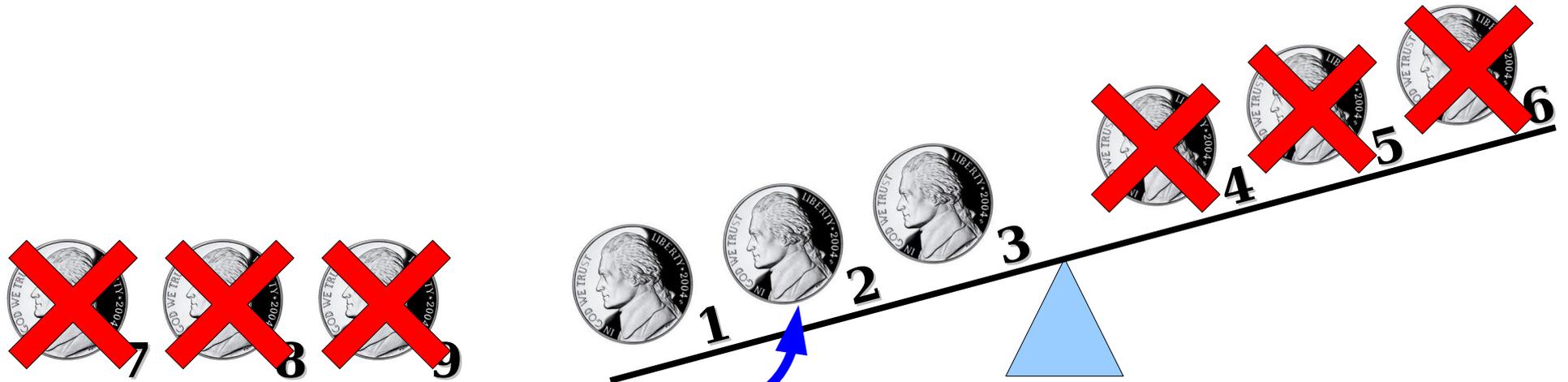
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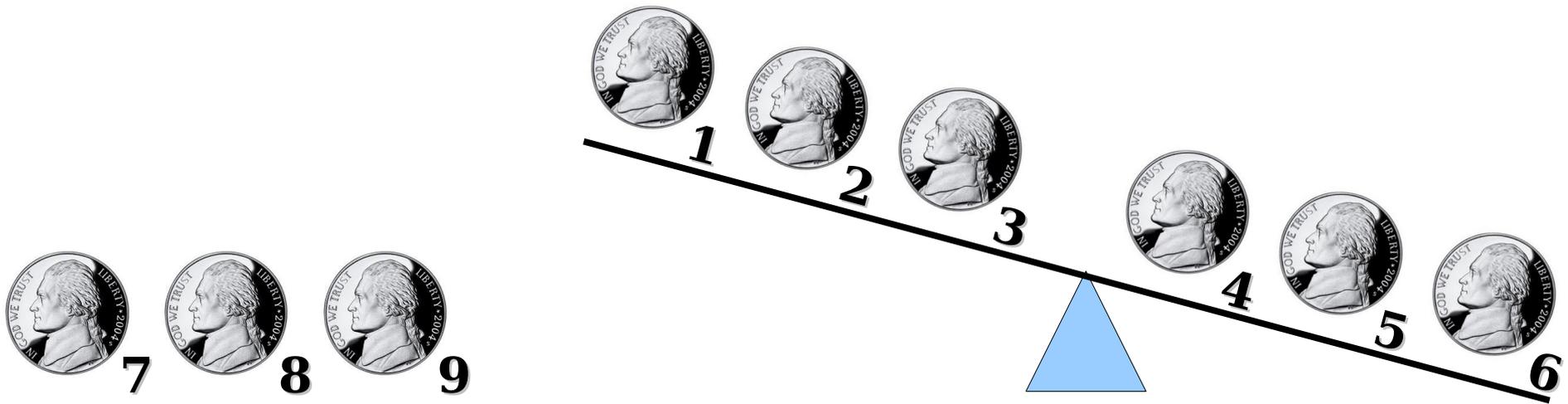


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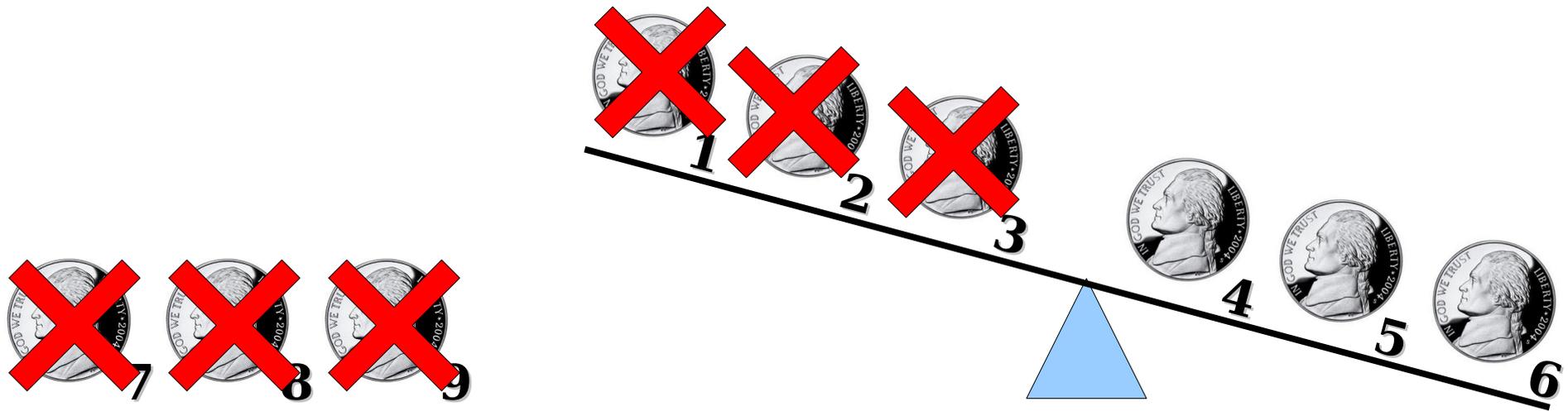


Now we have one weighing to find the counterfeit out of these three coins.

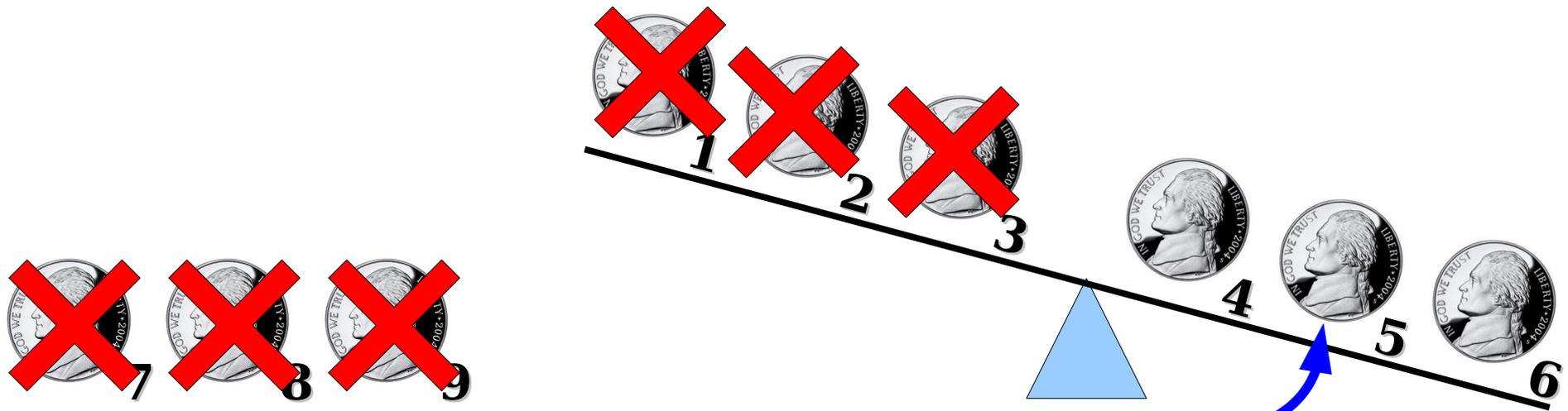
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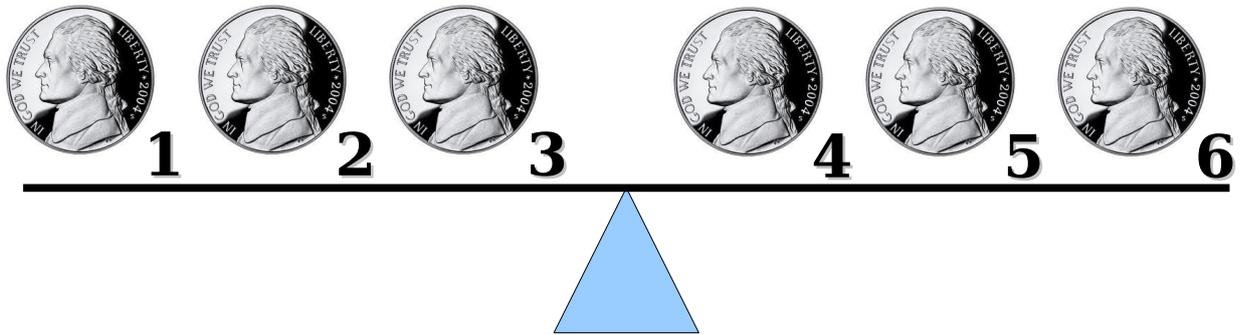


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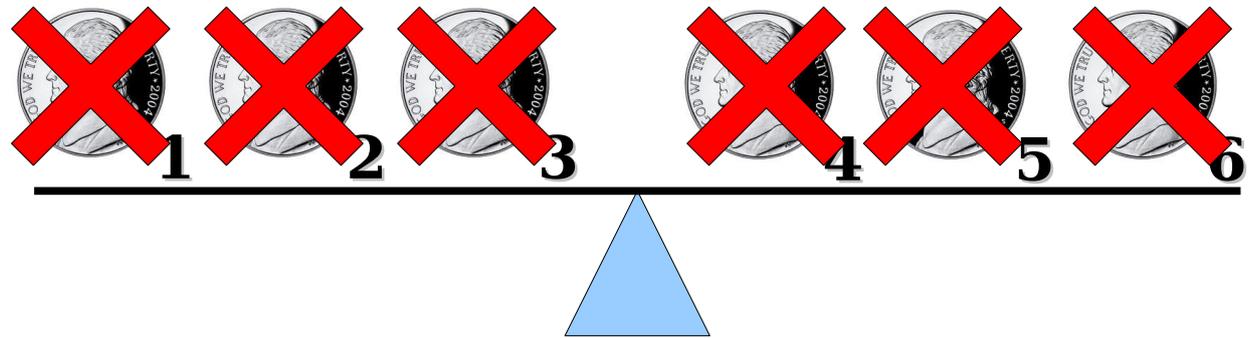


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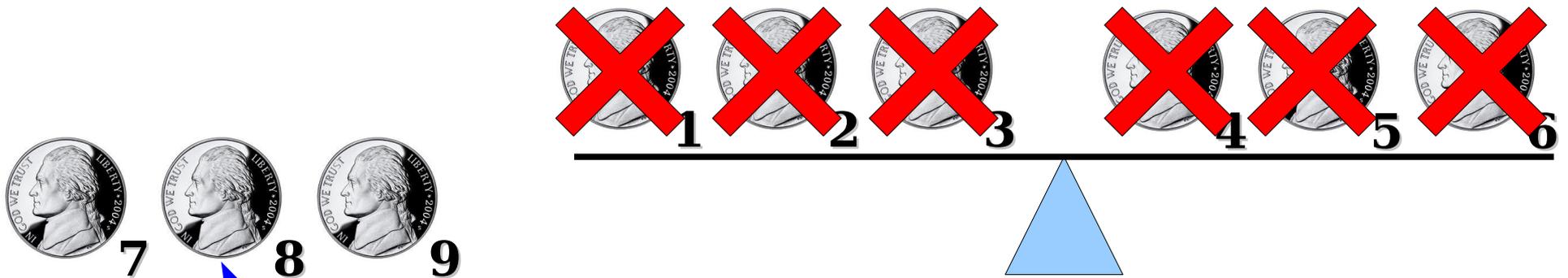
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Now we have one weighing to find the counterfeit out of these three coins.

Can we generalize this?

# A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - **One** coin, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of **three** coins.
- If we have two weighings, we can find the counterfeit out of **nine** coins.

So far, we have

$$\mathbf{1, 3, 9 = 3^0, 3^1, 3^2}$$

Does this pattern continue?

**Theorem:** If exactly one coin in a group of  $3^n$  coins is heavier than the rest, that coin can be found using only  $n$  weighings on a balance.

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers  $n$ , then tell them we're going to prove it by induction.

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The goal of this step is to prove

**“If  $P(k)$  is true, then  $P(k+1)$  is true.”**

So we ask the reader to choose an arbitrary  $k$ , assume that  $P(k)$  is true, then try to prove  $P(k+1)$ .

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Here, we explicitly state  $P(k+1)$ , which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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Here, we use our **inductive hypothesis** (the assumption that  $P(k)$  is true) to solve this simpler version of the overall problem.

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# Some Fun Problems

- Here's some nifty variants of this problem that you can work through:
  - Suppose that you have a group of coins where there's either exactly one heavier coin, or all coins weigh the same amount. If you only get  $k$  weighings, what's the largest number of coins where you can find the counterfeit or determine none exists?
  - What happens if the counterfeit can be either heavier or lighter than the other coins? What's the maximum number of coins where you can find the counterfeit if you have  $k$  weighings?
  - Can you find the counterfeit out of a group of more than  $3^k$  coins with  $k$  weighings?
  - Can you find the counterfeit out of any group of at most  $3^k$  coins with  $k$  weighings?

**Time-Out for Announcements!**

# Midterm Exam

- Our midterm is tomorrow night from 7PM – 10PM here in Bishop Auditorium.
  - ***You can do this***. Best of luck on the exam!
- Our TA Stanley will be holding a review session tonight from 7:30PM – 8:30PM in 200-107.
  - Come prepared with questions – you'll get more out of the session if you do.
- Some of our OH this week have moved; check the calendar for details.

Back to CS103!

# How Not To Induct

Something's Wrong...

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Where is the error in this proof?  
Answer at <https://cs103.stanford.edu/pollev>

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Where did we  
prove the base  
case?

Therefore,  $P(k + 1)$  is true, completing the induction. ■

When writing a proof by induction,  
***make sure to prove the base case!***  
Otherwise, your proof is incomplete!

Why did this work?

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$$2^0 + 2^1 + \dots + 2^{k-1} + 2^k$$

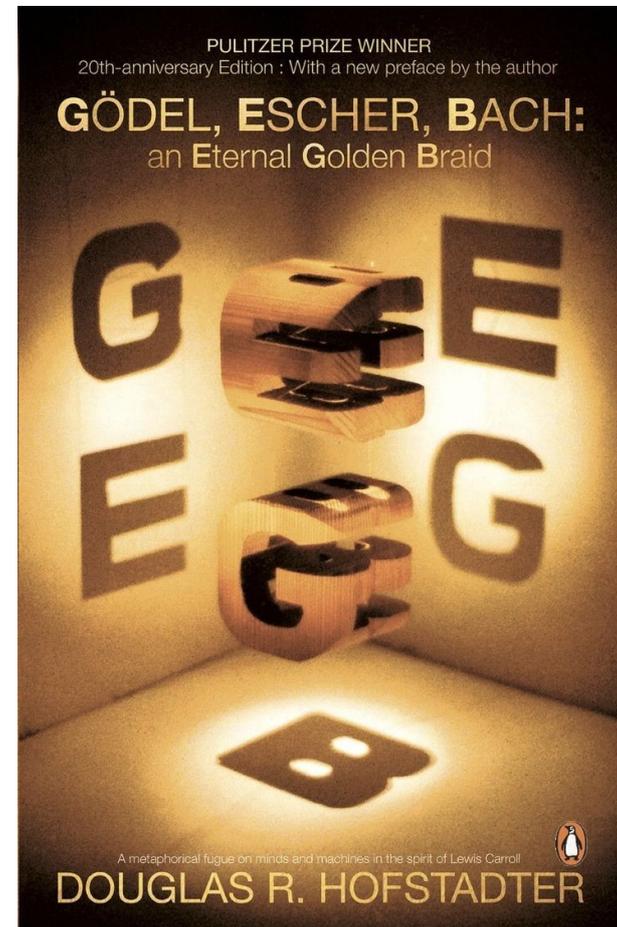
You can prove *anything* from a faulty assumption. This is called the *principle of explosion*.

Therefore,  $P(k + 1)$  is true, completing the induction. ■

# The MU Puzzle

# *Gödel, Escher, Bach: An Eternal Golden Braid*

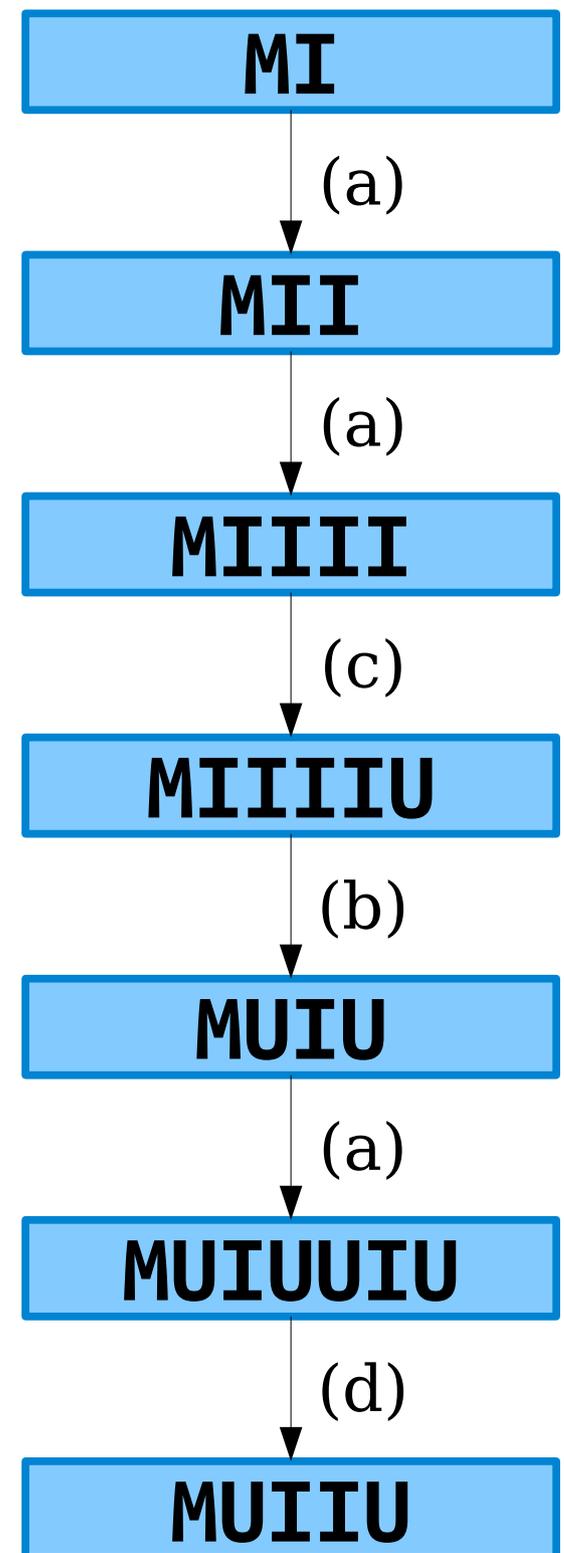
- Douglas Hofstadter, cognitive scientist at the University of Indiana, wrote this Pulitzer-Prize-winning mind trip of a book.
- It's a great read after you've finished CS103 - you'll see so many of the ideas we'll cover presented in a totally different way!



# The **MU** Puzzle

- Begin with the string **MI**.
- Repeatedly apply one of the following operations:
  - Double the contents of the string after the **M**: for example, **MIIU** becomes **MIIUIIU**, or **MI** becomes **MII**.
  - Replace **III** with **U**: **MIIII** becomes **MUI** or **MIU**.
  - Append **U** to the string if it ends in **I**: **MI** becomes **MIU**.
  - Remove any **UU**: **MUUU** becomes **MU**.
- **Question**: How do you transform **MI** to **MU**?

- (a) Double the string after an **M**.
- (b) Replace **III** with **U**.
- (c) Append **U**, if the string ends in **I**.
- (d) Delete **UU** from the string.



# Try It!

Starting with **MI**, apply these operations to make **MU**:

- (a) Double the string after an **M**.
- (b) Replace **III** with **U**.
- (c) Append **U**, if the string ends in **I**.
- (d) Delete **UU** from the string.

Not a single person in this room  
was able to solve this puzzle.

Are we even sure that there is a solution?

# Counting I's



# The Key Insight

- Initially, the number of **I**'s is *not* a multiple of three.
- To make **MU**, the number of **I**'s must end up as a multiple of three.
- Can we *ever* make the number of **I**'s a multiple of three?

**Lemma 1:** If  $n$  is an integer that is not a multiple of three, then  $n - 3$  is not a multiple of three.

**Proof:** By contrapositive; we'll prove that if  $n - 3$  is a multiple of three, then  $n$  is also a multiple of three. So pick some integer  $n$  where  $n - 3$  is a multiple of three. Thus there's an integer  $q$  where  $n - 3 = 3q$ . We then see  $n = 3q + 3 = 3(q + 1)$ , so there is an integer  $r$  (namely,  $q + 1$ ) where  $n = 3r$ . Thus  $n$  is a multiple of three. ■

**Lemma 2:** If  $n$  is an integer that is not a multiple of three, then  $2n$  is not a multiple of three.

**Proof:** Let  $n$  be an integer that isn't a multiple of three. We need to show that  $2n$  is not a multiple of three.

Since  $n$  is not a multiple of three, either  $n$  is congruent to 1 modulo 3 or  $n$  is congruent to 2 modulo 3. In the first case, we can write  $n = 3k + 1$  for some integer  $k$ , and we see that  $2n = 6k + 2 = 3(2k) + 2$  is congruent to 2 modulo 3. In the second, we write  $n = 3k + 2$  for some integer  $k$ , and note that  $2n = 6k + 4 = 3(2k + 1) + 1$ , which is congruent to 1 modulo 3. Either way,  $2n$  is not a multiple of three. ■

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Therefore, no sequence of  $k+1$  moves ends with a multiple of three **I**'s.

**Lemma:** No matter which moves are made, the number of **I**'s in the string never becomes multiple of three.

**Proof:** Let  $P(n)$  be the statement “after any  $n$  moves, the number of **I**'s in the string will not be multiple of three.” We will prove, by induction, that  $P(n)$  is true for all  $n \in \mathbb{N}$ , from which the theorem follows.

As a base case, we'll prove  $P(0)$ , that the number of **I**'s after 0 moves is not a multiple of three. After no moves, the string is **MI**, which has one **I** in it. Since one isn't a multiple of three,  $P(0)$  is true.

For our inductive step, suppose that  $P(k)$  is true for some arbitrary  $k \in \mathbb{N}$ . We'll prove  $P(k+1)$  is also true. Consider any sequence of  $k+1$  moves. Let  $r$  be the number of **I**'s in the string after the  $k$ th move. By our inductive hypothesis (that is,  $P(k)$ ), we know that  $r$  is not a multiple of three. Now, consider the four possible choices for the  $k+1^{\text{st}}$  move:

*Case 1:* Double the string after the **M**. After this, we will have  $2r$  **I**'s in the string, and from our lemma  $2r$  isn't a multiple of three.

*Case 2:* Replace **III** with **U**. After this, we will have  $r - 3$  **I**'s in the string, and by our lemma  $r - 3$  is not a multiple of three.

*Case 3:* Either append **U** or delete **UU**. This preserves the number of **I**'s in the string, so we don't have a multiple of three **I**'s at this point.

Therefore, no sequence of  $k+1$  moves ends with a multiple of three **I**'s. Thus  $P(k+1)$  is true, completing the induction.

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**Theorem:** The **MU** puzzle has no solution.

**Proof:** Assume for the sake of contradiction that the **MU** puzzle has a solution and that we can convert **MI** to **MU**. This would mean that at the very end, the number of **I**'s in the string must be zero, which is a multiple of three. However, we've just proven that the number of **I**'s in the string can never be a multiple of three.

We have reached a contradiction, so our assumption must have been wrong. Thus the **MU** puzzle has no solution. ■

# Algorithms and Loop Invariants

- The proof we just made had the form
  - “If  $P$  is true before we perform an action, it is true after we perform an action.”
- We could therefore conclude that after any series of actions of any length, if  $P$  was true beforehand, it is true now.
- In algorithmic analysis, this is called a ***loop invariant***.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!

# Next Time

- ***Variations on Induction***
  - Starting induction later.
  - Taking larger steps.
  - Complete induction.